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Sutherland-type Trigonometric Models, Trigonometric Invariants and Multivariate Polynomials

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Abstract. It is conjectured that any trigonometric Olshanetsky-Perelomov Hamiltonian written in *Fundamental* Trigonometric Invariants (FTI) as coordinates takes an algebraic form and preserves an infinite flag of spaces of polynomials. It is shown that try-and-guess variables which led to the algebraic form of trigonometric Olshanetsky-Perelomov Hamiltonians related to root spaces of the classical A_N, B_N, C_N, D_N, BC_N and exceptional G_2, F_4 Lie algebras are FTI. This conjecture is also confirmed for the trigonometric E_6 Olshanetsky-Perelomov Hamiltonian whose algebraic form is found with the use of FTI.

1. Introduction

About 30 years ago, Olshanetsky and Perelomov [**1**] (for a review, see [**2**]) discovered a remarkable family of quantum mechanical Hamiltonians with trigonometric potentials, which are associated to the crystallographic root spaces of the classical (A_N, B_N, C_N, D_N) and exceptional $(G_2, F_4, E_{6,7,8})$ Lie algebras. The Olshanetsky-Perelomov Hamiltonians have the property of complete integra- bility (the number of integrals of motion in involution is equal to the dimension of the configuration space) and that of exact solvability (the spectrum can be found explicitly, in a closed analytic form that is a second-degree polynomial in the quantum numbers). The Hamiltonian associated to a Lie algebra g of rank N, with root space Δ, is

$$H_\Delta = \frac{1}{2}\sum_{k=1}^{N}\left[-\frac{\partial^2}{\partial y_k^2}\right] + \frac{\beta^2}{8}\sum_{\alpha\in R_+} g^2_{|\alpha|}\frac{|\,\alpha|^{\,2}}{\sin^2\frac{\beta}{2}(\alpha\cdot y)}\ , \tag{1.1}$$

where R_+ is the set of positive roots of Δ, $\beta \in \mathbb{R}$ is a parameter introduced for convenience, $g^2_{|\alpha|} = \mu_{|\alpha|}(\mu_{|\alpha|} - 1)$ are coupling constants depending only on the root length, and $y = (y_1, y_2, \ldots, y_N)$ is the coordinate vector. If all roots are of the same length, then $g_{|\alpha|} = g$ (i.e. there is a single coupling constant). If the roots

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are of two different lengths, then for the long roots $g_{|\alpha|} = g_l$ and for the short ones $g_{|\alpha|} = g_s$ (i.e. there are two coupling constants). The configuration space here is the Weyl alcove of the root space (see [**2**]).

The ground state eigenfunction and its eigenvalue are

$$\Psi_0(y) \ = \ \prod_{\alpha\in R_+} \left|\sin\frac{\beta}{2}(\alpha\cdot y)\right|^{\mu_{|\alpha|}} \quad , \quad E_0 \ = \ \frac{\beta^2}{8}\rho^2 \ , \tag{1.2}$$

where $\rho = \sum_{\alpha\in R_+}\mu_{|\alpha|}\alpha$ is the so-called 'deformed Weyl vector' (see [**2**], eqs.(5.5), (6.7)). It is known that any eigenfunction Ψ has the form of (1.2) multiplied by a polynomial in exponential (trigonometric) coordinates, i.e. $\Psi = \Phi\Psi_0$ (see [**2**]). Such polynomials Φ are called (generalized) *Jack polynomials.* For connections between Jack polynomials, and the theory of special functions and orthogonal polynomials, see, e.g. [**3, 4**].

For future use, we make three definitions.

Definition 1. A multivariate linear differential operator is said to be in algebraic form if its coefficients are polynomials in the independent variable(s). It is called algebraic if by an appropriate change of the independent variable(s), it can be written in an algebraic form.

Definition 2. Consider a finite-dimensional (linear) space of multivariate polynomials defined as a linear span in the following way:

$$P^{(d)}_{n,\{\alpha\}} \ = \ \langle x_1^{p_1}x_2^{p_2}\dots x_d^{p_d} | 0 \le \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_d p_d \le n\rangle \quad ,$$

where the α's are positive integers and $n\in\mathbb{N}$. Its *characteristic vector* is the d-dimensional vector with components α_i[1]:

$$\vec{\alpha} = (\alpha_1, \alpha_2, \dots \alpha_d) \ . \tag{1.3}$$

For some characteristic vectors, the corresponding polynomial spaces may have a Lie-algebraic interpretation, in that they are the finite-dimensional representation spaces for some Lie algebra of (first-order) differential operators. For example, the spaces corresponding to $\vec{\alpha} = (1,\dots,1)$, indexed by n, are finite-dimensional representation spaces of the algebra $gl(d+1)$ of first-order differential operators.

Definition 3. Take the infinite set of spaces of multivariate polynomials $P_n \equiv P^{(d)}_{n,\{\alpha\}}$, $n\in\mathbb{N}$, defined as above, and order them by inclusion:

$$P_0 \subset P_1 \subset P_2 \subset \dots \subset P_n \subset \dots \ .$$

Such an object is called an *infinite flag (or filtration)*, and is denoted $P^{(d)}_{\{\alpha\}}$. If a linear differential operator preserves such an infinite flag, it is said to be *exactly-solvable.* It is evident that every such operator is algebraic (see [**5**]). If the spaces P_n can be viewed as the finite-dimensional representation spaces of some Lie algebra g, then g is called the *hidden algebra* of the exactly-solvable operator.

Any crystallographic root space Δ is characterized by its fundamental weights $w_a, a = 1, 2, \dots r$, where $r = \text{rank}(\Delta)$. One can take a fundamental weight w_a and

[1]We do not think that this notation will cause a confusion with positive roots.

generate its orbit Ω_a, by acting on it by all elements of the Weyl group of Δ. By averaging over this orbit, i.e. by computing

$$\tau_a(y) = \sum_{\omega \in \Omega_a} e^{i\beta(w\cdot y)} \ , \tag{1.4}$$

one obtains a trigonometric Weyl invariant for any specified $\beta \in \mathbb{R}$. For a given root space Δ and a fixed β, there thus exist r independent trigonometric Weyl invariants τ generated by r fundamental weights w_a. We shall call them *Fundamental* Trigonometric Invariants (FTI). For the theory of root spaces, see [**6**] and in a concise form, [**7**] or [**8**]. A brief description of FTI, under the name 'exponential invariants' appears in Bourbaki [**9**], (Ch.6, §3, p.194).

The goal of this paper is to show, for each of several Lie algebras g, (i) that the Jack polynomials arising from the eigenfunctions of the Hamiltonian (1.1), being rewritten in terms of FTI, remain polynomials in these invariants, (ii) that a similarity-transformed version of (1.1), namely $h \propto \Psi_0^{-1}(H - E_0)\Psi_0$, acting on the space of trigonometric invariants (i.e., the space of trigonometric orbits) is an operator in algebraic form, and (iii) that h preserves an infinite flag of spaces of polynomials, with a certain characteristic vector. Results are presented for the root spaces $A_N, BC_N, B_N, C_N, D_N, G_2, F_4$ and E_6. Although similar results might seem to be obtainable for E_7 and $E_{,8}$, an analysis of those root spaces is absent, mainly due to great technical complications.

2. The case $\Delta = A_N$

For the root space A_N, the Olshanetsky-Perelomov Hamiltonian (1.1) coincides with the Hamiltonian of the Sutherland model [**10**], and has the form

$$H_{\rm Suth} \ = \ -\frac{1}{2}\sum_{k=1}^{N+1}\frac{\partial^2}{\partial x_k^2} + \frac{g\beta^2}{4}\sum_{k<l}^{N+1}\frac{1}{\sin^2(\frac{\beta}{2}(x_k - x_l))} \ , \tag{2.1}$$

with the ground state eigenfunction

$$\Psi_0(x) \ = \ \prod_{i<j}^{N+1} \sin^\nu\left(\frac{\beta}{2}(x_i - x_j)\right) \ , \ g = \nu(\nu-1) > -\frac{1}{4} \ . \tag{2.2}$$

It describes a system of $(N+1)$ particles situated on a circle, with a pairwise interaction that is given by potential term in (2.1). For a review see [**12**].

In order to solve the eigenvalue problem for the Hamiltonian (2.1) let us introduce the Perelomov relative coordinates [**13**]

$$Y \ = \ \sum x_i \ , \ y_i \ = \ x_i - \frac{1}{N+1}Y \ , \ i = 1, \ldots, N+1 \ , \tag{2.3}$$

where Y is the center-of-mass coordinate, and the coordinates y_i are confined to the hyperplane

$$\sum_{i=1}^{N+1} y_i \ = \ 0 \ . \tag{2.4}$$

A transformation to Weyl-invariant periodic coordinates was introduced in [**12**]. It is

$$(x_1, x_2, \ldots x_{N+1}) \to \left(e^{i\beta Y}, \eta_n(x) = \sigma_n(e^{i\beta y(x)}) |\ n = 1, 2 \ldots N\right) \ , \tag{2.5}$$

where $\sigma_k(x) = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k},\ k=1,2,\ldots,N$ are the elementary symmetric polynomials, and by convention, $\sigma_0 = \sigma_{N+1} = 1$ and $\sigma_i = 0$ for $i < 0$ and $i > (N+1)$. It was shown that a similarity-transformed version of the Hamiltonian (2.1), namely $h_{A_N} = -\frac{2}{\beta^2}(\Psi_0)^{-1} H_{Suth}\,\Psi_0$, after separation of the center-of-mass motion ($Y = 0$) takes on the algebraic form

$$h_{\rm Suth} = \sum_{i,j=1}^{N} A_{ij}(\eta)\frac{\partial^2}{\partial\eta_i\partial\eta_j} + \sum_{i=1}^{N} B_i(\eta)\frac{\partial}{\partial\eta_i}\ , \tag{2.6}$$

where

$$A_{ij} \ = \ \frac{(N+1-i)\,j}{N+1}\,\eta_i\,\eta_j + \sum_{l\geq \max(1,j-i)} (j-i-2l)\,\eta_{i+l}\,\eta_{j-l} \qquad \text{at } i \geq j\ ,$$

$$A_{ji} \ = A_{ij}\ ,\ B_i \ = \ (\frac{1}{N+1}+\nu)\,i\,(N+1-i)\,\eta_i\ .$$

It can easily be checked [**12**] that the operator $h_{\rm Suth}$ preserves the infinite flag $P^{(N)}_{\{1,1,\ldots,1\}}$. This is in agreement with our general conjecture that the characteristic vector for the trigonometric model coincides with the minimal characteristic vector for corresponding rational model [**14**]. The operator h_{A_N} depends on a single parameter ν, linearly. The nodal structure of its eigenpolynomials (i.e. where they vanish) at fixed ν remains an open question.

STATEMENT 1. *For any n, one can find a fundamental weight w_a of the A_N root system for which $\eta_n = \tau_a$. Hence, the Weyl-invariant periodic coordinates $\eta_n, n = 1,2,\ldots,N$, defined in (2.5), coincide with the fundamental trigonometric invariants $\tau_a, a = 1,\ldots,N =$rank (A_N), defined in (1.4).*

To prove this statement, note that the fundamental weights of A_N can be written in terms of the canonical basis $e_1,\ldots,e_{N+1}$ of $\mathbb{R}^{N+1}$ as (see [**9**])

$$w_k = (e_1 + e_2 + \cdots + e_k) - \frac{k}{N+1}\sum_{j=1}^{N+1} e_j\ , \quad k = 1,N\ . \tag{2.7}$$

Hence, the orbit element related to a given fundamental weight reads at $Y = 0$ as

$$\exp(i\beta w_k\cdot y) = \exp\left(i\beta\sum_{j=1}^{k} y_j\right) = \prod_{j=1}^{k}\exp(i\beta y_j)\ . \tag{2.8}$$

Since the Weyl group for A_N is a symmetric group S_{N+1} that permutes the vectors e_j, the averaging of (2.8) over this group gives for the FTI exactly $\sigma_k(\exp(i\beta y))$. It is worth noting that there exists a symmetry [**12**]: the involution $\beta \leftrightarrow -\beta$ corresponds to $\eta_i \leftrightarrow \eta_{N+1-i}$ (see (2.8)). Since the original Hamiltonian depends on β^2, this leads to certain relations between the coefficients A_{ij} and $B_i \leftrightarrow B_{N+1-i}$.

Let us consider the algebra $gl(N+1)$ realized by the first order differential operators

$$J_i^- = \frac{\partial}{\partial \tau_i}, \qquad i = 1, 2 \dots N \,, \tag{2.9a}$$

$$J_{ij}{}^0 = \tau_i \frac{\partial}{\partial \tau_j}, \qquad i, j = 1, 2 \dots N \,, \tag{2.9b}$$

$$J^0 = \sum_{i=1}^{d} \tau_i \frac{\partial}{\partial \tau_i} - n \,, \tag{2.9c}$$

$$J_i^+ = \tau_i J^0 = \tau_i \left(\sum_{j=1}^{d} \tau_j \frac{\partial}{\partial \tau_j} - n \right), \quad i = 1, 2 \dots N \,, \tag{2.9d}$$

where n is any number. If in (2.9), n is non-negative integer, the generators (2.9) will have a common invariant subspace $P^{(N)}_{n,\{1,1,\dots,1\}}$, on which they act irreducibly. Hence the infinite flag $P^{(N)}_{\{1,1,\dots,1\}}$ is made of irreducible finite-dimensional representation spaces of the algebra gl_{N+1}. If the raising generators J^+ are excluded, the remaining generators will form the maximal affine subalgebra of the $gl(N+1)$ algebra. It is evident that the generators J^0, J^- preserve $P^{(N)}_{\{1,1,\dots,1\}}$. It can be proved that $h_{\rm Suth}$, given in (2.6) can be rewritten in terms of the generators J^0, J^-. Therefore, gl_{N+1} is the hidden algebra of the Sutherland model.

3. The case $\Delta = BC_N$ (including $\Delta = B_N, C_N, D_N$)

For the root space BC_N the Olshanetsky-Perelomov Hamiltonian (1.1) has the form

$$\begin{aligned} H_{BC_N} = & -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial {x_i}^2} + \frac{g\beta^2}{4}\sum_{i<j}^{N}\left[\frac{1}{\sin^2\left(\frac{\beta}{2}(x_i-x_j)\right)} + \frac{1}{\sin^2\left(\frac{\beta}{2}(x_i+x_j)\right)}\right] \\ & + \frac{g_2\beta^2}{2}\sum_{i=1}^{N}\frac{1}{\sin^2\beta x_i} + \frac{g_3\beta^2}{8}\sum_{i=1}^{N}\frac{1}{\sin^2\frac{\beta x_i}{2}} \,. \end{aligned} \tag{3.1}$$

with the ground state eigenfunction

$$\Psi_0 = \prod_{i<j}^{N} |\sin(\frac{\beta}{2}(x_i-x_j))|^{\nu} |\sin(\frac{\beta}{2}(x_i+x_j))|^{\nu} \prod_{i=1}^{N} |\sin(\beta x_i)|^{\nu_2} |\sin(\frac{\beta}{2}x_i)|^{\nu_3} \,, \tag{3.2}$$

where $g = \nu(\nu-1) > -1/4$, $g_2 = \nu_2(\nu_2-1) > -1/4$, $g_3 = \nu_3(\nu_3+2\nu_2-1) > -1/4$. From the general BC_N Hamiltonian (3.1) the B_N, C_N and D_N cases are obtained by specializing as follows:

- B_N case: $\nu_2 = 0$,
- C_N case: $\nu_3 = 0$,
- D_N case: $\nu_2 = \nu_3 = 0$.

In order to solve the eigenvalue problem for the BC_N Hamiltonian (3.1), let us perform a change of variables to Weyl-invariant periodic coordinates [**15**], i.e.,

$$(x_1, x_2, \dots x_N) \to \big(\eta_n(x) = \sigma_n(\cos\beta x)|_{\ n=1,2\dots N}\big) \tag{3.3}$$

where $\sigma_k(x)$ $, k = 1, 2, \ldots, N$ are the elementary symmetric polynomials, with $\sigma_0 = 1$. The similarity-transformed Hamiltonian (3.1):

$$h_{BC_N} = -\frac{2}{\beta^2}(\Psi_0)^{-1}\, H_{BC_N}\, \Psi_0$$

with Ψ_0 given by (3.2), has the form (see [**15**])

$$h_{BC_N} \;=\; \sum_{i,j=1}^{N} A_{ij}(\eta)\frac{\partial^2}{\partial\eta_i\partial\eta_j} + \sum_{i=1}^{N} B_i(\eta)\frac{\partial}{\partial\eta_i}\ , \tag{3.4}$$

where

$$\begin{aligned}
A_{ij} \;&=\; N\,\eta_{i-1}\,\eta_{j-1} - \sum_{l\geq 0}\Big[(i-l)\,\eta_{i-l}\,\eta_{j+l} + (l+j-1)\,\eta_{i-l-1}\,\eta_{j+l-1} \\
&\quad -(i-2-l)\,\eta_{i-2-l}\,\eta_{j+l} - (l+j+1)\,\eta_{i-l-1}\,\eta_{j+l+1}\Big]\ , \\
B_i \;&=\; \nu_3(i-N-1)\,\eta_{i-1} - \Big[1+\nu(2N-i-1)+2\nu_2+\nu_3\Big]\,i\,\eta_i \\
&\quad -\nu(N-i+1)(N-i+2)\eta_{i-2}\ .
\end{aligned}$$

Here $\eta_0 = 1$ and by convention, $\eta_i = 0$ for $i < 0$ and $i > N$.

It can be easily checked that the operator h_{BC_N} preserves the infinite flag $P^{(N)}_{\{1,1,\ldots,1\}}$, similarly to the case of the A_N Hamiltonian. This is in agreement with our conjecture that the characteristic vector for a trigonometric model always coincides with the minimal characteristic vector for the corresponding rational model [**14**].

The BC_N model depends on three parameters ν, ν_2, ν_3, and the nodal structure of the eigenpolynomials (i.e. where they vanish) at fixed ν's remains an open question.

STATEMENT 2: *For any n, one can find a fundamental weight w_a of the C_N root system for which $\eta_n = f_a\tau_a$, where f_a is a constant. Hence, the Weyl-invariant periodic coordinates $\eta_n, n = 1, \ldots, N$ (3.3) coincide with the fundamental trigonometric invariants $\tau_a, a = 1, \ldots,$* N=rank (C_N), defined in *(1.4)*, up to numerical factors. The coefficients in (3.4) are changed accordingly.

Indeed, the element of the k-th orbit related to the fundamental weight $w_k = (e_1 + e_2 + \cdots + e_k)$ $,\quad k = 1, 2, \ldots N)$ (see [**9**]), looks like

$$\exp(i\beta w_k\cdot x) = \prod_{j=1}^{k}\exp(i\beta x_j)\ . \tag{3.5}$$

The Weyl group for C_N root space is a semidirect product of a permutation group S_N acting on the vectors e_i and a group $(\mathbb{Z}/2\mathbb{Z})^N$ that acts as $e_j \mapsto (\pm 1)_j e_j$. Averaging (3.5) over the second group action gives $2^k\prod_{j=1}^{k}\cos(\beta x_j)$, and averaging over permutations gives $\sigma_k(\cos(\beta x))$, up to a common multiplicative factor.

REMARK 1: The set of C_N trigonometric invariants of the form (1.4) is characterized by the smallest common period, in comparison with the set of the B_N or D_N trigonometric invariants. In general, any C_N trigonometric invariant can be rewritten as a polynomial either in B_N or in D_N invariants.

REMARK 2: Neither the B_N Hamiltonian ($g_2 = 0$ in (3.1)) nor the D_N Hamiltonian ($g_2 = g_3 = 0$ in (3.1)) takes on an algebraic form in terms of the B_N or D_N

trigonometric invariants, respectively. However, both B_N and D_N Hamiltonians take on an algebraic form in terms of the C_N trigonometric invariants.

It can be shown that the operator $h_{\rm BC_N}$ of (3.4) can be rewritten in terms of the generators J^0, J^- of (2.9) (see [**15**]), similarly to the A_N model; cf. (2.6). Therefore, gl_{N+1} is the hidden algebra of the BC_N model.

4. The case $\Delta = G_2$

The Olshanetsky-Perelomov Hamiltonian (1.1) for the root space G_2 has the form

$$H_{\rm G_2} = -\frac{1}{2}\sum_{k=1}^{3}\frac{\partial^2}{\partial x_k^2} + \frac{g\beta^2}{4}\sum_{k<l}^{3}\frac{1}{\sin^2(\frac{\beta}{2}(x_k-x_l))}$$

$$+\frac{g_1\beta^2}{4}\sum_{k<l;k,l\neq m}^{3}\frac{1}{\sin^2(\frac{\beta}{2}(x_k+x_l-2x_m))}\ , \tag{4.1}$$

where $g=\nu(\nu-1)>-\frac{1}{4}$ and $g_1=3\mu(\mu-1)>-\frac{3}{4}$ are the coupling constants associated with two-body and three-body interactions, respectively. From a physical point of view, the Hamiltonian (4.1) describes a system of three identical particles that are situated on a circle. The ground state eigenfunction is

$$\Psi_0(x) = (\Delta^{(trig)}(x))^{\nu}(\Delta_1^{(trig)}(x))^{\mu}\ , \tag{4.2}$$

where $\Delta^{(trig)}(x),\ \Delta_1^{(trig)}(x)$ are the trigonometric analogies of the Vandermonde determinant and are defined by

$$\Delta^{(trig)}(x) \ = \ \prod_{k<l}^{3}|\sin\frac{\beta}{2}(x_k-x_l)|\,, \tag{4.3a}$$

$$\Delta_1^{(trig)}(x) \ = \ \prod_{k<l;k,l\neq m}^{3}|\sin\frac{\beta}{2}(x_k+x_l-2x_m)|\ . \tag{4.3b}$$

In order to solve the eigenvalue problem for the Hamiltonian (4.1), let us introduce the Perelomov coordinates $Y, y_i,\ i=1,2,3$ as in (2.3). The relative coordinates y_i are constrained by $y_1+y_2+y_3\ =\ 0$. It was shown in [**16**] that after separating the center-of-mass coordinate in (4.1), and introducing the Weyl-invariant periodic coordinates

$$\eta_1 \ = \ \frac{-2}{\beta^2}\left[\sin^2\frac{\beta}{2}(y_1-y_2)+\sin^2\frac{\beta}{2}(y_2-y_3)+\sin^2\frac{\beta}{2}(y_3-y_1)\right], \tag{4.4a}$$

$$\eta_2 \ = \ \frac{4}{\beta^6}\left[\sin\beta(y_1-y_2)+\sin\beta(y_2-y_3)+\sin\beta(y_3-y_1)\right]^2\ , \tag{4.4b}$$

the similarity-transformed Hamiltonian $h_{G_2}\ =\ -2(\Psi_0)^{-1}H_{G_2}\Psi_0$ takes, after the transformation to new coordinates $(x_1,x_2,x_3)\to\big(Y,\ \eta_1,\ \eta_2\big)$ the algebraic form

$$h_{G_2}(\eta) \ = \ -\left(2\eta_1+\frac{\beta^2}{2}\eta_1^2-\frac{\beta^4}{24}\eta_2\right)\partial^2_{\eta_1\eta_1} - \left(12+\frac{8\beta^2}{3}\eta_1\right)\eta_2\partial^2_{\eta_1\eta_2}$$

$$+\left(\frac{8}{3}\eta_1^2\eta_2-2\beta^2\eta_2^2\right)\partial^2_{\eta_2\eta_2} - \left\{2[1+3(\mu+\nu)]+\frac{2}{3}(1+3\mu+4\nu)\beta^2\eta_1\right\}\partial_{\eta_1}$$

$$+\left\{\frac{4}{3}(1+2\nu)\eta_1^2-[\frac{7}{3}+4(\mu+\nu)]\beta^2\eta_2\right\}\partial_{\eta_2}\ . \tag{4.5}$$

It can easily be checked that the operator h_{G_2} preserves the infinite flag $P^{(2)}_{\{1,2\}}$. This is in agreement with our general conjecture that the characteristic vector for a trigonometric model coincides with the minimal characteristic vector for the corresponding rational model [**14**].

The root space G_2 has two fundamental weights; namely, $a_1 = e_3 - e_1$ and $a_2 = -e_1 - e_2 + 2e_3$. Averaging over the orbits generated by a_1 and a_2 (as in (1.4)), we end up with the explicit FTI

$$\tau_1 \quad = \quad 2[\cos(\beta(y_1-y_2))+\cos(\beta(2y_1+y_2))+\cos(\beta(2y_1+y_2))]\ , \tag{4.6a}$$

$$\tau_2 \quad = \quad 2[\cos(3\beta y_1)+\cos(3\beta y_2)+\cos(3\beta(y_1+y_2))]\ . \tag{4.6b}$$

One can easily verify a connection between $\eta_{1,2}$ and $\tau_{1,2}$:

$$\eta_1=\frac{1}{2\beta^2}(\tau_1-6)\ ,\ \eta_2=\frac{1}{\beta^6}(4\tau_2-\tau_1^2+12)\ . \tag{4.7}$$

The transformation (4.7) does not alter the infinite flag $P^{(2)}_{\{1,2\}}$. Changing the variables in (4.5) from η's to τ's we end up with

$$h_{G_2}(\tau)\ =\ \frac{1}{8\beta^2}h_{G_2}\ =$$

$$\left(4+\tau_1+\frac{\tau_2}{3}-\frac{\tau_1^2}{3}\right)\partial^2_{\tau_1\tau_1}-\left(12+4\tau_2+\tau_1\tau_2-2\tau_1^2\right)\partial^2_{\tau_1\tau_2}-\left(9\tau_1+3\tau_2+3\tau_1\tau_2+\tau_2^2-\tau_1^3\right)\partial^2_{\tau_2\tau_2}$$

$$+\left[2\nu-\frac{1+3\mu+4\nu}{3}\tau_1\right]\partial_{\tau_1}-\left[3(2\mu+\nu)+(1+2\mu+2\nu)\tau_2+\frac{\nu}{12}\tau_1^2\right]\partial_{\tau_2}\ . \tag{4.8}$$

A straightforward analysis confirms a conclusion that the operator $h_{G_2}(\tau)$ preserves the infinite flag $P^{(2)}_{\{1,2\}}$. This is not a surprising result, since the transformation (4.7) maps each subspace in $P^{(2)}_{\{1,2\}}$ to itself.

The G_2 model depends on two parameters ν, μ, and the nodal structure of eigenpolynomials (i.e. where they vanish) at fixed ν and μ remains an open question.

Let us consider an infinite-dimensional Lie algebra of the differential operators generated by the following eight operators

$$\left\{\begin{array}{ll} L^1=\partial_{\tau_1}\ , & L^2=\tau_1\partial_{\tau_1}-\frac{n}{3}\ , \\ L^3=2\tau_2\partial_{\tau_2}-\frac{n}{3}\ , & L^4=\tau_1^2\partial_{\tau_1}+2\tau_1\tau_2\partial_{\tau_2}-n\tau_1\ , \\ L^5=\partial_{\tau_2}\ , & L^6=\tau_1\partial_{\tau_2}\ , \\ L^7=\tau_1^2\partial_{\tau_2}\ , & T=\tau_2\partial^2_{\tau_1\tau_1}\ . \end{array}\right. \tag{4.9}$$

This algebra was introduced for the first time in [**16**], being called $g^{(2)}$. The generators $L^i, i=1,\dots,7$, generate a subalgebra of the form $gl_2 \ltimes R^3$. For each $n \in \mathbb{N}$, the generators (4.9) have the common invariant subspace

$$P^{(2)}_{n,\{1,2\}}\ =\ \langle \tau_1^{n_1}\tau_2^{n_2} \,|\, 0 \le (n_1+2n_2) \le n\rangle\ , \tag{4.10}$$

(cf. Definition 2), on which they act irreducibly [2]. The common invariant spaces $P^{(2)}_{n,\{1,2\}}, n \in \mathbb{N}$ form the infinite flag $P^{(2)}_{\{1,2\}}$. If the generator L^4 is excluded, the remaining generators preserve $P^{(2)}_{\{1,2\}}$.

It can be easily shown that the operator (4.8) can be rewritten in terms of the generators of the algebra $g^{(2)}$, in the $n = 0$ case, as

$$\begin{aligned} h_{G_2}(\tau) \ &= \ 4L^1L^1 + L^2L^1 + \frac{1}{3}T - \frac{1}{3}L^2L^2 - 12L^1L^5 - 2L^1L^3 \\ &-\frac{1}{2}L^2L^3 + 2L^6L^2 - 9L^5L^6 - \frac{3}{2}L^3L^5 - \frac{3}{2}L^3L^6 - \frac{1}{4}L^3L^3 + L^6L^7 \\ &+ 2\nu L^1 - \frac{3\mu+4\nu}{3}L^2 - 3(2\mu+\nu)L^5 - \frac{1+4\mu+4\nu}{4}L^3 - \frac{\nu}{12}L^7 \ . \end{aligned} \tag{4.11}$$

In this representation, the generator L^4 is absent. Hence, the algebra $g^{(2)}$ is the hidden algebra of the G_2 trigonometric model.

5. The case $\Delta = F_4$

The trigonometric F_4 model is defined by the Olshanetsky-Perelomov Hamiltonian (1.1) for the root space F_4 [3],

$$\begin{aligned} H_{\mathrm{F}_4} = -\frac{1}{2}\sum_{i=1}^{4}\partial_{x_i}^2 + \frac{g\beta^2}{4} \sum_{j>i}\left(\frac{1}{\sin^2\frac{\beta(x_i-x_j)}{2}} + \frac{1}{\sin^2\frac{\beta(x_i+x_j)}{2}}\right) \\ + g_1\beta^2 \sum_{i=1}^{4}\frac{1}{\sin^2\beta x_i} + g_1\beta^2 \sum_{\nu's=0,1}\frac{1}{\sin^2\frac{\beta[x_1+(-1)^{\nu_2}x_2+(-1)^{\nu_3}x_3+(-1)^{\nu_4}x_4]}{2}} \ . \end{aligned} \tag{5.1}$$

where β is a parameter and $g, g_1 > -1/4$ are the coupling constants. If $g_1 = 0$ the Hamiltonian (5.1) degenerates to that of the trigonometric D_4 model (i.e. (3.1) at $N = 4$ and $g_2 = g_3 = 0$). The trigonometric F_4 model is completely integrable, for arbitrary values of the coupling constants g, g_1. It describes a quantum particle in a four-dimensional space.

The ground state of the Hamiltonian (5.1) is

$$\Psi_0 = \Delta_-^{\nu}(\beta)\Delta_+^{\nu}(\beta)\Delta_0^{\mu}(\beta)\Delta^{\mu}(\beta) \ , \tag{5.2}$$

where degrees ν, μ are related with the coupling constants by

$$g = \nu(\nu-1) \ , \qquad g_1 = \frac{1}{2}\mu(\mu-1) \ , \tag{5.3}$$

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[2] It is also worth mentioning that at $n = 0$, the algebra $gl_2 \ltimes R^3$ becomes an algebra of vector fields, acting on a 2-Hirzebruch surface, Σ_2, and the modules are the sections of holomorphic line bundles over this surface (see [**17**] and references therein).

[3] Actually, this form corresponds to the representation of the F_4-Hamiltonian for the dual root space (see a discussion in [**18**]). It was chosen for convenience in making calculations. Calculations with the F_4-Hamiltonian defined for the root space turned to be more complicated.

and

$$\text{(5.4a)} \quad \Delta_{\pm}(\beta) \;=\; \prod_{j<i}^{4} \sin\frac{\beta(x_i \pm x_j)}{2}\ ,$$

$$\text{(5.4b)} \quad \Delta_0(\beta) \;=\; \prod_{i=1}^{4} \sin\beta x_i\ ,$$

$$\text{(5.4c)} \quad \Delta(\beta) \;=\; \prod_{\nu' s} \sin\frac{\beta\,[x_1 + (-1)^{-\nu_2}x_2 + (-1)^{-\nu_3}x_3 + (-1)^{-\nu_4}x_4]}{2}\ .$$

The ground state eigenvalue is

$$\text{(5.5)} \qquad E_0 \;=\; (7\nu^2 + 14\mu^2 + 18\nu\mu)\beta^2\ .$$

In [**18**], as a result of a series of intelligent guesses, after rather sophisticated and tedious analysis there were found surprisingly simple variables in terms of which the similarity-transformed version of the Hamiltonian (5.1),

$$\text{(5.6)} \qquad h_{\mathrm{F}_4} \;=\; -2(\Psi_0)^{-1}(H_{\mathrm{F}_4} - E_0)(\Psi_0)\ ,$$

takes the form of an algebraic operator. This operator was derived explicitly (see [**18**]). It preserved the infinite flag $P^{(4)}_{\{1,2,2,3\}}$. This is in agreement with our conjecture that the characteristic vector for a trigonometric model always coincides with the minimal characteristic vector for the corresponding rational model [**14**]. The explicit expressions for the abovementioned variables are[4]

$$\text{(5.7a)} \qquad \eta_2 \;=\; \widetilde{\eta}_1 - \frac{\beta^2}{6}\widetilde{\eta}_2\ ,$$

$$\text{(5.7b)} \qquad \eta_6 \;=\; \widetilde{\eta}_3 - \frac{1}{6}\,\widetilde{\eta}_1\,\widetilde{\eta}_2 - \frac{\beta^2}{2}(\widetilde{\eta}_4 - \frac{1}{36}\widetilde{\eta}_2^2),$$

$$\text{(5.7c)} \qquad \eta_8 \;=\; \widetilde{\eta}_4 - \frac{1}{4}\,\widetilde{\eta}_1\,\widetilde{\eta}_3 + \frac{1}{12}\,\widetilde{\eta}_2^2,$$

$$\text{(5.7d)} \qquad \eta_{12} \;=\; \widetilde{\eta}_4\,\widetilde{\eta}_2 - \frac{1}{36}\,\widetilde{\eta}_2^3 - \frac{3}{8}\,\widetilde{\eta}_3^2 + \frac{1}{8}\,\widetilde{\eta}_1\,\widetilde{\eta}_2\,\widetilde{\eta}_3 - \frac{3}{8}\,\widetilde{\eta}_1^2\,\widetilde{\eta}_4\ ,$$

where $\tilde{\eta}$'s are the elementary symmetric polynomials

$$\text{(5.8)} \qquad \widetilde{\eta}_i \;=\; \sigma_i\left(\frac{4\sin^2\frac{\beta}{2}x}{\beta^2}\right)\ ,\ i = 1, 2, 3, 4\ ,$$

(For a definition of σ's, see (2.5)). Below we shall show that there is nothing mysterious in the variables (5.7a); they can easily be obtained from the FTI of (1.4).

Associated with the algebra F_4 are a root space and a dual-root space. From a technical point of view, it is more convenient to work in the dual root space. Thus, we shall define the FTI by averaging over orbits in the dual root space. In the dual root space, the fundamental weights

$$\text{(5.9)} \quad a_1 = e_3 + e_4\ ,\ a_2 = 2e_4\ ,\ a_3 = e_2 + e_3 + 2e_4\ ,\ a_4 = e_1 + e_2 + e_3 + 3e_4\ ,$$

[4] In the limit β tends to zero the variables η's go to the polynomial invariants of the F_4 root space which are classified following the degrees of the F_4 algebra $(2, 6, 8, 12)$. Numbering of η's variables reflects this fact.

generate orbits with lengths equal to $(24, 24, 96, 96)$, respectively. Averaging over the orbits, as in (1.4), we define FTI that we denote by $\tau_{1,2,3,4}$, respectively. After some algebra one finds an explicit relation between η's and τ's:

(5.10a) $\eta_2 = -\frac{1}{24}\frac{\tau_1 - 24}{\beta^2}$,

(5.10b) $\eta_6 = \frac{1}{4608}\frac{\tau_1^2 + 24\tau_1 - 36\tau_2 - 288}{\beta^6}$,

(5.10c) $\eta_8 = \frac{1}{3072}\frac{\tau_1^2 - 12\tau_1 - 3\tau_3}{\beta^8}$,

(5.10d) $\eta_{12} = -\frac{1}{294912}\frac{2\tau_1^3 + 72\tau_1^2 - 9\tau_1\tau_3 - 864\tau_1 - 324\tau_2 - 216\tau_3 + 27\tau_4 - 1728}{\beta^{12}}$.

It is evident that this transformation leads to an algebraic form for h_{F_4}; and in fact, this algebraic form preserves the infinite flag $P^{(4)}_{\{1,2,2,3\}}$. The form of this operator in terms of the trigonometric invariants (the τ's) is the following

$$h_{F_4}(\tau) \equiv \frac{1}{4\beta^2}h_{F_4} = \sum_{i,j=1}^{4} A_{ij}(\tau)\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{i=1}^{4} B_i(\tau)\frac{\partial}{\partial\tau_i}\ , \tag{5.11}$$

where

$$A_{11} = -2\tau_1^2 + 24\tau_1 + 12\tau_2 + 2\tau_3 + 96\ ,\ A_{12} = -2\tau_1\tau_2 + 24\tau_1 + 6\tau_3\ ,$$
$$A_{13} = 24\tau_1^2 + 8\tau_1\tau_2 - 3\tau_1\tau_3 - 192\tau_1 - 84\tau_2 - 48\tau_3 + 3\tau_4 - 576\ ,$$
$$A_{14} = 8\tau_1\tau_2 - 4\tau_1\tau_4 + 4\tau_2\tau_3 - 96\tau_1 - 24\tau_3\ ,$$
$$A_{22} = 24\tau_1^2 - 4\tau_2^2 - 192\tau_1 - 96\tau_2 - 48\tau_3 + 4\tau_4 - 384\ ,$$

$$A_{23} = -48\tau_1^2 - 8\tau_1\tau_2 + 6\tau_1\tau_3 - 4\tau_2\tau_3 + 480\tau_1 + 216\tau_2 + 120\tau_3 - 18\tau_4 + 1152\ ,$$

$$\begin{aligned} A_{24} = {} & -48\tau_1^3 - 8\tau_1^2\tau_2 \\ & +192\tau_1^2 + 208\tau_1\tau_2 + 144\tau_1\tau_3 - 12\tau_1\tau_4 + 24\tau_2^2 + 16\tau_2\tau_3 - 6\tau_2\tau_4 + 6\tau_3^2 \\ & +3072\tau_1 + 960\tau_2 + 576\tau_3 - 96\tau_4 + 4608\ , \end{aligned}$$

$$\begin{aligned} A_{33} = {} & 24\tau_1^3 + 8\tau_1^2\tau_2 - 192\tau_1^2 - 120\tau_1\tau_2 - 72\tau_1\tau_3 + 2\tau_1\tau_4 - 8\tau_2\tau_3 - 6\tau_3^2 \\ & -768\tau_1 - 96\tau_2 - 96\tau_3 + 24\tau_4\ , \end{aligned}$$

$$\begin{aligned} A_{34} = {} & 4\tau_1\tau_2\tau_3 - 32\tau_1^2\tau_2 + 192\tau_1^2 + 288\tau_1\tau_2 - 24\tau_1\tau_3 - 16\tau_1\tau_4 \\ & +144\tau_2^2 + 64\tau_2\tau_3 - 12\tau_2\tau_4 - 8\tau_3\tau_4 - 1920\tau_1 - 96\tau_2 - 480\tau_3 + 72\tau_4 - 4608\ , \end{aligned}$$

$$\begin{aligned} A_{44} = {} & -32\tau_1^3\tau_2 - 384\tau_1^3 - 192\tau_1^2\tau_2 - 16\tau_1^2\tau_4 + 96\tau_1\tau_2^2 + 4\tau_2\tau_3^2 \\ & +96\tau_1\tau_2\tau_3 - 8\tau_1\tau_2\tau_4 + 2688\tau_1^2 + 1728\tau_2^2 + 48\tau_3^2 - 12\tau_4^2 \\ & +5760\tau_1\tau_2 + 1152\tau_1\tau_3 + 32\tau_1\tau_4 + 1024\tau_2\tau_3 - 48\tau_2\tau_4 + 32\tau_3\tau_4 \\ & +15360\tau_1 + 12288\tau_2 + 2304\tau_3 + 192\tau_4 + 18432\ , \end{aligned}$$

and

$$B_1 = -2(1 + 6\,\mu + 5\,\nu)\tau_1 - 48\,\nu\ ,$$
$$B_2 = -12\,\nu\,\tau_1 - 4(1 + 5\,\mu + 3\,\nu)\,\tau_2 - 96\,\mu\ ,$$
$$B_3 = -48(\mu + \nu)\,\tau_1 - 24\,\nu\,\tau_2 - 6(1 + 4\,\mu + 3\,\nu)\,\tau_3\ ,$$

$$\begin{aligned} B_4 &= -48\,\mu\,\tau_1^2 - 8\,\nu\,\tau_1\tau_2 + 48\,(8\,\mu + \nu)\,\tau_1 + 48\,(4\,\mu - \nu)\,\tau_2 \\ &\quad +96\,\mu\,\tau_3 - 12\,(1 + 3\,\mu + 2\,\nu)\,\tau_4 + 1152\,\mu\;. \end{aligned}$$

A straightforward analysis confirms the conclusion that the operator $h_{F_4}(\tau)$ preserves the infinite flag $P^{(4)}_{\{1,2,2,3\}}$. This is not a surprising result, since the transformation (5.10) maps each subspace in $P^{(4)}_{\{1,2,2,3\}}$ to itself.

The F_4 model depends on two parameters ν, μ, and the nodal structure of the eigenpolynomials (i.e. where they vanish) at fixed ν and μ remains an open question.

This Hamiltonian can be written in terms of the generators of an infinitedimensional algebra of differential operators $f^{(4)}$ generated by 49 operators, which admits finite-dimensional representations in terms of inhomogeneous polynomials in four variables (see [**18**]). Among those 49 operators there are 22 differential operators of the first order, 22 of the second and 5 the of third.

6. The case $\Delta = E_6$

The Hamiltonian of the trigonometric E_6 model is built using the root system of the E_6 algebra (see (1.1)). A convenient way to represent the Hamiltonian in coordinate form is to use an 8-dimensional space with coordinates $x_1, x_2, \ldots x_8$ imposing two constraints: $x_7 = x_6$, $x_8 = -x_6$. In terms of these coordinates,

$$H_{E_6} = -\frac{1}{2}\Delta^{(8)} + \frac{g\beta^2}{4}\sum_{j<i=1}^{5}\left[\frac{1}{\sin^2\frac{\beta}{2}(x_i+x_j)} + \frac{1}{\sin^2\frac{\beta}{2}(x_i-x_j)}\right] \tag{6.1}$$
$$+\frac{g\beta^2}{4}\sum_{\{\nu_j\}}\frac{1}{\left[\sin^2\dfrac{\beta}{4}\left(-x_8+x_7+x_6-\sum_{j=1}^5(-1)^{\nu_j}x_j\right)\right]}\;,$$

the second summation being one over quintuples $\{\nu_j\}$ where each $\nu_j = 0, 1$, and $\sum_{j=1}^5 \nu_j$ is even. Here $g = \nu(\nu - 1) > -1/4$ is the coupling constant. The configuration space is the principal E_6 Weyl alcove.

In order to resolve the constraints, we introduce new variables:

$$\begin{aligned} y_i &= x_i\;, \quad i = 1\ldots 5 \\ y_6 &= x_6 + x_7 - x_8\;, \qquad \text{(with the constraint } y_6 = 3x_6\text{)}, \\ y_7 &= x_6 - x_7\;, \qquad \text{(with the constraint } y_7 = 0\text{)}, \\ y_8 &= x_6 + x_8\;, \qquad \text{(with the constraint } y_8 = 0\text{)}. \end{aligned} \tag{6.2}$$

In terms of these, the Laplacian has the representation

$$\Delta^{(8)} = \Delta^{(5)}_y + 3\frac{\partial^2}{\partial y_6^2} + 2\left[\frac{\partial^2}{\partial y_7^2} + \frac{\partial^2}{\partial y_8^2} + \frac{\partial^2}{\partial y_7 \partial y_8}\right]\;, \tag{6.3}$$

while the potential part of (6.1) depends on $y_1 \ldots y_6$ only:

$$\begin{aligned} V &= \frac{g\beta^2}{4}\sum_{j<i=1}^{5}\left[\frac{1}{\sin^2\frac{\beta}{2}(y_i+y_j)^2} + \frac{1}{\sin^2\frac{\beta}{2}(y_i-y_j)}\right] \\ &\quad + \frac{g\beta^2}{4}\sum_{\nu_j,j=1}^{5}\frac{1}{\left[\sin^2\frac{\beta}{4}\left(y_6-\sum_{j=1}^5(-1)^{\nu_j}y_j\right)\right]}\;. \end{aligned} \tag{6.4}$$

In this formalism, imposing the constraints requires that one should study only eigenfunctions having no dependence on y_7, y_8. Hence, the $y_{7,8}$-dependent part of the Laplacian standing in square brackets in (6.3) can simply be dropped.

The ground state eigenfunction and its eigenvalue are

$$\Psi_0 = (\Delta_+^{(5)}\Delta_-^{(5)})^\nu \Delta_{E_6}^\nu \ , \ E_0 = 39\beta^2\nu^2 \ , \tag{6.5}$$

where

$$\Delta_\pm^{(5)} = \prod_{j<i=1}^{5} \sin\frac{\beta}{2}(y_i \pm y_j) \ , \tag{6.6a}$$

$$\Delta_{E_6} = \prod_{\{\nu_j\}} \sin\frac{\beta}{4}(y_6 + \sum_{j=1}^{5}(-1)^{\nu_j} y_j) \ . \tag{6.6b}$$

The main object of our study is the similarity-transformed version of the Hamiltonian (6.1), with the ground state eigenfunction (6.5) taken as a factor, i.e.

$$h_{\mathrm{E}_6} = -\frac{8}{\beta^2}(\Psi_0)^{-1}(H_{\mathrm{E}_6} - E_0)(\Psi_0) \ , \tag{6.7}$$

where E_0 is given by (6.5).

The E_6 root space is characterized by 6 fundamental weights, which generate orbits of lengths ranging from 27 to 720. Let us introduce an ordering of the fundamental trigonometric invariants τ_a defined by (1.4), namely:

orbit Variable	weight vector	orbit size
τ_1	$-2e_6$	27
τ_2	$e_5 - e_6$	27
τ_3	$e_4 + e_5 - 2e_6$	216
τ_4	$-\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5) - \frac{5}{2}e_6$	216
τ_5	$\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5) - \frac{3}{2}e_6$	72
τ_6	$e_3 + e_4 + e_5 - 3e_6$	720

The pairs of variables (τ_1, τ_2) and (τ_3, τ_4) are complex conjugates. The orbit variables have certain transformation properties under the involution $\beta \to -\beta$: $\tau_{1,3}(-\beta) = \tau_{2,4}(\beta)$, while $\tau_{5,6}$ remain unchanged, i.e. are invariant. Since the Hamiltonian is invariant under $\beta \to -\beta$, after converting to τ-variables it should be invariant under the simultaneous interchange $\tau_1 \leftrightarrow \tau_2, \tau_3 \leftrightarrow \tau_4$.

After very lengthy, truly cumbersome and tedious calculations which probably will be published elsewhere, one can show that the similarity-transformed Hamiltonian (6.7), in terms of the above trigonometric invariants (τ-variables), takes on an algebraic form. This is the following:

$$h_{E_6} = \sum_{i,j=1}^{6} A_{ij}(\tau)\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{i=1}^{6} B_i(\tau)\frac{\partial}{\partial\tau_i} \ , \tag{6.8}$$

where

$$A_{11} = -\frac{4\tau_1^2}{3} + 20\tau_2 + 2\tau_4 \, , \quad A_{12} = -\frac{2\tau_1\tau_2}{3} + 6\tau_5 + 54 \, ,$$

$$A_{13} = -\frac{4\tau_1\tau_3}{3} + 5\tau_2\tau_5 - 32\tau_2 - 5\tau_4 \, , \quad A_{14} = 16\tau_1\tau_2 - \frac{5\tau_1\tau_4}{3} - 51\tau_5 + 3\tau_6 - 432 \, ,$$

$$A_{15} = -\tau_1\tau_5 + 32\tau_1 + 5\tau_3 \, , \quad A_{16} = 10\tau_1\tau_5 - 2\tau_1\tau_6 - 64\tau_2^2 - 4\tau_2\tau_4 + 4\tau_3\tau_5 + 384\tau_1 + 78\tau_3 \, ,$$

$$A_{22} = -\frac{4\tau_2^2}{3} + 20\tau_1 + 2\tau_3\,, \quad A_{23} = 16\tau_1\tau_2 - \frac{5\tau_2\tau_3}{3} - 51\tau_5 + 3\tau_6 - 432\,,$$
$$A_{24} = -\frac{4\tau_2\tau_4}{3} + 5\tau_1\tau_5 - 32\tau_1 - 5\tau_3\,, \quad A_{25} = -\tau_2\tau_5 + 32\tau_2 + 5\tau_4\,,$$
$$A_{26} = -64\tau_1^2 - 4\tau_1\tau_3 + 10\tau_2\tau_5 - 2\tau_2\tau_6 + 4\tau_4\tau_5 + 384\tau_2 + 78\tau_4\,,$$
$$A_{33} = 16\tau_1\tau_2^2 - 64\tau_1^2 - 24\tau_1\tau_3 - 36\tau_2\tau_5 + 2\tau_2\tau_6 - \frac{10\tau_3^2}{3} + 4\tau_4\tau_5 - 208\tau_2 + 8\tau_4\,,$$
$$A_{34} = 4\tau_1\tau_2\tau_5 - 176\tau_1\tau_2 - \frac{8\tau_3\tau_4}{3} + 6\tau_5^2 + 528\tau_5 - 42\tau_6 + 3888\,,$$
$$A_{35} = 32\tau_2^2 + 4\tau_2\tau_4 - 2\tau_3\tau_5 - 224\tau_1 - 44\tau_3\,,$$
$$\begin{aligned} A_{36} \;=\;& -128\tau_1^2\tau_2 + 5\tau_1\tau_5^2 + 3\tau_2\tau_4\tau_5 + 320\tau_1\tau_5 - 32\tau_1\tau_6 \\ & +576\tau_2^2 + 104\tau_2\tau_4 - \tau_3\tau_5 - 4\tau_3\tau_6 + 5\tau_4^2 + 192\tau_1 - 312\tau_3\,, \end{aligned}$$
$$A_{44} = 16\tau_1^2\tau_2 - 36\tau_1\tau_5 + 2\tau_1\tau_6 - 64\tau_2^2 - 24\tau_2\tau_4 + 4\tau_3\tau_5 - \frac{10\tau_4^2}{3} - 208\tau_1 + 8\tau_3\,,$$
$$A_{45} = 32\tau_1^2 + 4\tau_1\tau_3 - 2\tau_4\tau_5 - 224\tau_2 - 44\tau_4\,,$$
$$\begin{aligned} A_{46} \;=\;& -128\tau_1\tau_2^2 + 3\tau_1\tau_3\tau_5 + 5\tau_2\tau_5^2 + 576\tau_1^2 + 104\tau_1\tau_3 + 320\tau_2\tau_5 \\ & -32\tau_2\tau_6 + 5\tau_3^2 - \tau_4\tau_5 - 4\tau_4\tau_6 + 192\tau_2 - 312\tau_4\,, \end{aligned}$$
$$A_{55} = 16\tau_1\tau_2 - 2\tau_5^2 - 36\tau_5 + 2\tau_6 - 144\,,$$
$$A_{56} = -96\tau_1\tau_2 + 3\tau_3\tau_4 + 15\tau_5^2 - 3\tau_5\tau_6 + 216\tau_5 - 12\tau_6 + 864\,,$$
$$\begin{aligned} A_{66} \;=\;& -64\tau_1^2\tau_2^2 + 4\tau_1\tau_2\tau_5^2 + 256\tau_1^3 + 32\tau_1^2\tau_3 + 80\tau_1\tau_2\tau_5 - 24\tau_1\tau_2\tau_6 + 4\tau_1\tau_3^2 \\ & -16\tau_1\tau_4\tau_5 + 256\tau_2^3 + 32\tau_2^2\tau_4 - 16\tau_2\tau_3\tau_5 + 4\tau_2\tau_4^2 + 2\tau_3\tau_4\tau_5 + 6\tau_5^3 - 2112\tau_1\tau_2 \\ & -96\tau_1\tau_4 - 96\tau_2\tau_3 + 84\tau_3\tau_4 + 216\tau_5^2 + 36\tau_5\tau_6 - 6\tau_6^2 + 2592\tau_5 + 288\tau_6 + 10368\,, \end{aligned}$$

and

$$B_1 = -\frac{4(6+\nu)}{3}\tau_1\ ,\ \ B_2 = -\frac{4(6+\nu)}{3}\tau_2\ ,$$
$$\begin{aligned} B_3 \;=\;& -\frac{1}{18}[(1-\nu)(\tau_1{}^2 + 5\tau_1\,\tau_2 + 9\tau_2{}^2 - 15\tau_2 - 54\tau_4 - 45\tau_5 - 405) \\ & +30(13+3\nu)\tau_1 + (171+49\nu)\tau_3]\ , \end{aligned}$$
$$\begin{aligned} B_4 \;=\;& -\frac{1}{18}[(1-\nu)(9\tau_1{}^2 + 5\tau_1\,\tau_2 + \tau_2{}^2 - 15\tau_1 - 54\tau_3 - 45\tau_5 - 405) \\ & +30(13+3\nu)\tau_2 + (171+49\nu)\tau_4]\ , \end{aligned}$$
$$\begin{aligned} B_5 \;=\;& -\frac{1}{108}(1-\nu)[2(\tau_1{}^2 + \tau_1\,\tau_2 + \tau_2{}^2) - 30(\tau_1+\tau_2) - 3(\tau_3+\tau_4)] \\ & -\frac{13(5+\nu)}{6}\tau_5 - \frac{3}{2}(47+\nu)\ , \end{aligned}$$
$$\begin{aligned} B_6 \;=\;& \frac{(1-\nu)}{108}[2(\tau_1{}^3 + 2\tau_1{}^2\tau_2 + 2\tau_1\tau_2{}^2 + \tau_2{}^3) - 3(26\tau_1{}^2 + 9\tau_1\tau_3 + 11\tau_1\tau_4 + 12\tau_1\tau_5 \\ & +26\tau_2{}^2 + 11\tau_2\,\tau_3 + 9\tau_2\tau_4 + 12\tau_2\tau_5) + 3438(\tau_1+\tau_2) + 522(\tau_3+\tau_4)] \\ & -\frac{(43+29\nu)}{3}\tau_1\tau_2 + (23+61\nu)\tau_5 - (20+7\nu)\tau_6 + 108(1+5\nu)\ . \end{aligned}$$

After some analysis, one finds that the operator (6.8) preserves the infinite flag $P^{(6)}_{\{1,1,2,2,2,3\}}$. Its characteristic vector $\vec{\alpha} = (1,1,2,2,2,3)$ coincides with the minimal characteristic vector for the corresponding rational model [**14**]. This confirms our

conjecture that the characteristic vector for a trigonometric model always coincides with the minimal characteristic vector for the corresponding rational model.

It is worth mentioning that the operator (6.8) has a symmetry with respect to FTI (orbit variables) generated by orbits of the same length (see above); i.e., $\tau_1 \leftrightarrow \tau_2, \tau_3 \leftrightarrow \tau_4$. Under this involution

$$A_{12} \leftrightarrow A_{12}\ ,\ A_{13} \leftrightarrow A_{24}\ ,\ A_{14} \leftrightarrow A_{23}\ ,\ A_{15} \leftrightarrow A_{25}\ ,\ A_{16} \leftrightarrow A_{26}\ ,\ A_{33} \leftrightarrow A_{44}\ ,$$

$$A_{34} \leftrightarrow A_{34}\ ,\ A_{35} \leftrightarrow A_{45}\ ,\ A_{36} \leftrightarrow A_{46}\ ,\ A_{55} \leftrightarrow A_{55}\ ,\ A_{56} \leftrightarrow A_{56}\ ,\ A_{66} \leftrightarrow A_{66}\ ,$$

$$B_1 \leftrightarrow B_2\ ,\ B_3 \leftrightarrow B_4\ ,\ B_5 \leftrightarrow B_5\ ,\ B_6 \leftrightarrow B_6\ .$$

The E_6 model depends on the parameter ν, and the nodal structure of eigenpolynomials (i.e. where they vanish) at fixed ν remains an open question.

7. Summary and conclusions

Weyl-invariant coordinates leading to the algebraic forms of the trigonometric Olshanetsky-Perelomov Hamiltonians associated to the crystallographic root spaces A_N, BC_N, G_2, F_4 were found (in [**12**], [**15**], [**16**], [**18**], respectively) in a manner that was specific to each problem. In this paper, we have shown that the fundamental trigonometric invariants (FTI), if used as coordinates, provide a systematic way of reducing the trigonometric Hamiltonians associated to A_N, B_N, C_N, D_N, BC_N, and G_2, F_4, E_6, to algebraic form. The eigenfunctions of the trigonometric Hamiltonians (i.e., the Jack polynomials) remain polynomials in the FTI. The use of FTI enabled us to find an algebraic form of the Hamiltonian associated to E_6, which did not seem feasible at all, in the past. The calculations in this paper were based on a straightforward change of variables from Cartesian coordinates to FTI. Actually, there are clear indications of the existence of a representation-theoretic formalism that may allow such results to be derived more rapidly and elegantly [**7, 8, 19**].

Each of the Olshanetsky-Perelomov Hamiltonians, in algebraic form, preserves an infinite flag of polynomial spaces, with a characteristic vector $\vec{\alpha}$ that coincides with the minimal characteristic vector for the corresponding rational model (cf. [**14**]). It is worth noting that the matrices A_{ij} in the algebraic form Hamiltonians given explicitly in Eqs. (2.6), (3.4), (4.8), (5.11), (6.8), with polynomial entries, correspond to flat-space metrics, in the sense that the associated Riemann tensor vanishes. The change of variables in the corresponding Laplace-Beltrami operator, from FTI to Cartesian coordinates, transforms these metrics to diagonal form.

It should be stressed that each Hamiltonian of the form (1.1) is completely integrable. This implies the existence of a number of operators (the 'higher Hamiltonians') which commute with it and which are in involution. It is evident that these commuting operators take on an algebraic form after a gauge rotation (with the corresponding ground state eigenfunction as a gauge factor), and a change of variables from Cartesian coordinates to the FTI, i.e., to the τ's. Although both the original Hamiltonian (1.1) and the FTI (1.4) depend on the real parameter β, the resulting algebraic forms are β-independent. This fact yields a non-trivial connection between the algebraic operators of trigonometric models and the corresponding rational models. In practice, the connection is made in the following way: (i) take the set of FTI, specially ordered; and (ii) subtract from each a certain nonlinear combination of the other FTI, in such a way that as $\beta \to 0$, one obtains the polynomial Weyl invariants, which in the rational case lead to an algebraic operator, preserving a minimal flag. Surprisingly, if one changes variables to these

transformed FTI, the operator, expressed in terms of them, remains in algebraic form. This makes it possible to derive the algebraic operator of the rational model by taking the $\beta \to 0$ limit [**14**]. The significance of the ordering of the FTI, on which this procedure depends, is not clear to the authors. An analysis similar to the analysis of this paper has not yet been presented for the case of the trigonometric Olshanetsky-Perelomov Hamiltonians related to the exceptional root spaces E_7 and E_8. We conjecture that in these cases as well, the FTI taken as coordinates will yield an algebraic form for the Hamiltonian, and that the infinite flag of polynomial spaces with the same characteristic vector as in the corresponding rational model will be preserved. In concluding, we mention that the existence of algebraic forms of Olshanetsky- Perelomov Hamiltonians makes possible the study of their perturbations by purely algebraic means: one can develop a perturbation theory in which all corrections are found by linear-algebraic methods [**20**]. It also gives a hint that quasi-exactly-solvable generalizations of the Olshanetsky-Perelomov Hamiltonians may exist.

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